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ORIGINAL PAPER

Some differential properties of Pólya's cycle indicators

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Abstract Given a permutation group G acting on a nonempty finite set X ($|X| = n$), let $Z(G; z_1, z_2, \ldots, z_n)$ denote the *Pólya's cycle indicator of* G, and G_x the *stabilizer of an element* $x \in X$. We derive certain differential relationships for cycle indicators and, in particular, prove (Proposition 5) that if G is *transitive* on X, then for any $x \in X$

$$z_1 \frac{\partial}{\partial z_1} Z(G; z_1, z_2, \ldots, z_n) = Z(G_x; z_1, z_2, \ldots, z_n).$$

Keywords Polya's cycle indicator · Orbit · Transitive group · Stabilizer · Differentiation

1 Preliminaries

The famous *cycle index, or cycle indicator* [1–3], and its generalizations [4–6] have many applications in mathematics and other areas [1–8]. This continues to give impetus to new studies of its analytical properties. Herein, we derive some differential relationships between the cycle indicator of a transitive permutation group and the cycle indicator of the stabilizer of a point. For this purpose, we need to introduce some notation.

Let X ($|X| = n$) be a nonempty finite set and G be a permutation group acting on X ($G \cong \mathrm{Aut}X$). Further, let $Z(G; z_1, z_2, \ldots, z_n)$ denote the cycle indicator of G,

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and $G\backslash\backslash X$ the set of orbits induced by G on X. Also, G_x denotes the stabilizer of an element $x \in X$ (*i. e.*, $gx = x, \forall g \in G_x$), and X_g stands for the set of elements fixed by a permutation $g \in G$.

Recall that *Pólya's cycle index* is defined as follows [1–3]:

$$Z(G; z_1, z_2, \dots, z_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{i \,\big|\, |G|} z_i^{\alpha_i(g)}, \tag{1}$$

where $|G|$ is the cardinality of a group G; z_i's are weight-indeterminates; $\alpha_i(g)$ is the number of orbits of length i induced by a permutation $g \in G$; and the sum runs over all elements of G, while the product is taken over all divisors i of $|G|$.

Seeking to keep to a concise treatment of the subject, we cite herein just less-common but useful properties of the cycle index. The first one is from Exercise 2.1.1 on p. 59 in [3], *viz.*:

Proposition 1 *Let G be a permutation group acting transitively on a nonempty finite set X* $(|X| = n)$. *Then*

$$|G_x\backslash\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X_g|^2. \tag{2}$$

The cycle indicator $Z(G; z_1, z_2, \dots, z_n)$ also has some differential properties. From Proposition 10 on p. 125 of [4], we have:

Proposition 2 *Let* $Z(G; z_1, z_2, \dots, z_n)$ *be Pólya's cycle indicator. Then*

$$z_1 \frac{\partial}{\partial z_1} Z(G; z_1, z_2, \dots, z_n) = \frac{1}{|G|} \sum_{x \in X} |G_x| Z(G_x; z_1, z_2, \dots, z_n), \tag{3}$$

where the summation runs over all the set X.

The next statement (the Proposition 11 on p. 125 of [4]) symbolizes a converse passage from Pólya's theorem to the Cauchy–Frobenius lemma (see [3,4], also often termed "Burnside's lemma"), which was used to derive this theorem. *Viz.*:

Proposition 3 *Let* $Z(G; z_1, z_2, \dots, z_n)$ *be Pólya's cycle indicator. Then*

$$|G\backslash\backslash X| = \frac{\partial}{\partial z_1} Z(G; z_1, z_2, \dots, z_n) \Big|_{z_i = 1 \quad (i \in [1,n])}. \tag{4}$$

In a similar vein, we can cite (the Proposition 12 on p. 125 of [4]):

Proposition 4 *Let* $Z(G; z_1, z_2, \dots, z_n)$ *be Pólya's cycle indicator. Then*

$$\frac{1}{n} \sum_{s=1}^{n} s z_s \frac{\partial}{\partial z_s} Z(G; z_1, z_2, \dots, z_n) = Z(G; z_1, z_2, \dots, z_n). \tag{5}$$

Now, fueled with relevant knowledge, we turn to derivation of new combinatorial relationships.

2 Main part

In this section, we demonstrate several refinements of propositions given above, obtained for the case when G acts transitively on X. We begin with our basic statement:

Proposition 5 *Let G be a permutation group acting transitively on a nonempty finite set X* ($|X| = n$). *Then*

$$z_1 \frac{\partial}{\partial z_1} Z(G; z_1, z_2, \ldots, z_n) = Z(G_x; z_1, z_2, \ldots z_n). \tag{6}$$

Proof Since G acts transitively on X, $|G| = |X| \cdot |G_x|$, and the stabilizers of all elements $x \in X$ are isomorphic. Therefore, all summands on the R. H. S. of (3) give the same contribution to the x-sum. So, this sum may be replaced by the product of $|X| \left(= \frac{|G|}{|G_x|}\right)$ and any of its summands. Whence the result follows. □

Now, we can deduce also the following corollary:

Corollary 5.1 *Let G be a permutation group acting transitively on a nonempty finite set X* ($|X| = n$). *Then*

$$|G_x \backslash\backslash X| = \left\{ \frac{\partial}{\partial z_1} \left[z_1 \frac{\partial}{\partial z_1} Z(G; z_1, z_2, \ldots, z_n) \right] \right\} \Big|_{z_i = 1 \quad (i \in [1,n])}. \tag{7}$$

Proof Differentiate with respect to z_1 the expressions on both sides of (6), and then set $z_i = 1$ ($i \in [1, n]$). By applying Proposition 5 [with transposing the R. H. S. and L. H. S. of (6)], we immediately arrive at the proof. □

The next corollary is slightly less evident:

Corollary 5.2 *Let G be a permutation group acting transitively on a nonempty finite set X* ($|X| = n$). *Then*

$$|G_x \backslash\backslash X| = 1 + \left[\frac{\partial^2}{\partial z_1^2} Z(G; z_1, z_2, \ldots, z_n) \right] \Big|_{z_i = 1 \quad (i \in [1,n])}. \tag{8}$$

Proof Obviously, the differentiation of the L. H. S. of (3) with respect to z_1 gives

$$\frac{\partial}{\partial z_1} \left[z_1 \frac{\partial}{\partial z_1} Z(G; z_1, z_2, \ldots, z_n) \right] = \frac{\partial}{\partial z_1} Z(G; z_1, z_2, \ldots, z_n) + z_1 \frac{\partial^2}{\partial {z_1}^2} Z(G; z_1, z_2, \ldots, z_n). \tag{9}$$

By applying Proposition 3 to the first summand on the R. H. S. of (9) and taking into account that for a transitive group G $|G \backslash\backslash X| = 1$, one can easily confirm the occurrence of 1 on the R. H. S. of (8). By virtue of Proposition 3, the differentiation of the R. H. S. of (3) with respect to z_1 and subsequent calculation of the obtained expression under $z_i = 1$ $(i \in [1, n])$ results in $|G_x \backslash\backslash X|$. Considering all these steps together immediately leads to the proof. □

The last corollary is due to Proposition 1 and Corollary 5.1, *viz.*:

Corollary 5.3 *Let G be a permutation group acting transitively on a nonempty finite set X* $(|X| = n)$*. Then*

$$\frac{1}{|G|}\sum_{g\in G}|X_g|^2 = \left\{\frac{\partial}{\partial z_1}\left[z_1\frac{\partial}{\partial z_1}Z(G; z_1, z_2, \ldots, z_n)\right]\right\}\Bigg|_{z_i=1 \quad (i\in[1,n])}. \tag{10}$$

Consider a simple example. Let $\Gamma = (V; E)$ $(|V| = 6; |E| = 5)$ be a graph obtained by attaching a pendant edge to the middle vertex of the path P_5. This is a hydrogen-suppressed molecular graph of methyl-3-pentane, whose set V of vertices corresponds to carbon atoms, and set E of edges corresponds to C–C bonds; while hydrogen atoms are not considered. There are four orbits of symmetry-equivalent vertices in Γ: one is represented by two distal vertices, another by two their adjacent neighbors, one by a central vertex, and one by a pendant vertex. Therefore, a molecule of methyl-3-pentane may produce four different monoradicals, which may be used as substituents to other molecules. Often, chemists use also substituted radicals, on their own. For instance, we may consider, as substituents, halogens F, Cl, Br, I and allow maximum one halogen atom to be attached to each carbon atom (if a halogen substitutes for a hydrogen atom). We want to count the number of all such possibilities to derive halogen-substituted radicals together with unsubstituted ones (neglecting stereoisomers). First, consider a solution using the sum of cycle indicators enumerating substitutional isomers of four different rooted subgraphs of Γ. We have the following sum:

$$Y(V; z_1, z_2) = 2(z_1^6) + 2\left[(1/2)(z_1^6 + z_1^2 z_2^2)\right] = 3z_1^6 + z_1^2 z_2^2. \tag{11}$$

The same solution can be obtained using the differentiation of the cycle indicator counting substitutional isomers of the pristine molecule of methyl-3-pentane:

$$Y(V; z_1, z_2) = z_1\frac{\partial}{\partial z_1}Z(V; z_1, z_2) = z_1\frac{\partial}{\partial z_1}\left[(1/2)(z_1^6 + z_1^2 z_2^2)\right] = 3z_1^6 + z_1^2 z_2^2, \tag{12}$$

which is due to Proposition 2. In order to obtain a numeric result, we make a substitution $z_1 = z_2 = 5$ to $Y(V; z_1, z_2)$, where 5 reckons 4 sorts of halogens plus 1 unsubstituted hydrogen; this gives: $3 \times 5^6 + 5^2 \times 5^2 = 47500$. Although (12) here is a not so advantageous over (11), in case of bigger molecular graphs, using the differentiation of the basic cycle indicator $Z(V; z_1, z_2, \ldots, z_n)$ with respect to z_1 may give an essential economy in calculation of the generating function $Y(V; z_1, z_2, \ldots, z_n)$.

In a broader context, the differentiation of the generalized cycle indicators with respect to various variables was applied by Rosenfeld and Klein in [5,6], who introduced special hybrids of the cycle indicator and graph polynomials (such as the matching, characteristic, permanental, etc. ones). They offered a technique which allows to enumerate substitutional isomers with restrictive mutual positions of ligands. The book [8] contains numerous examples in which differential operators are employed for obtaining cycle indicators and other generating functions. These examples may tell the reader about practical problems which are solved using such methods. The theory of species of structures [8] is a field where the differential operators are very effectively used for performing various combinatorial tasks.

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